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LETTER TO THE EDITOR

On the self-similarities of a model set

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Abstract. We present a method to determine certain self-similarities of model sets.

The collection of spaces and mappings

$$\begin{array}{ccccc} \mathbb{R}^n & \xleftarrow{\pi_1} & \mathbb{R}^n \times \mathbb{R}^{k-n} & \xrightarrow{\pi_2} & \mathbb{R}^{k-n} \\ & & \cup & & \\ & & \mathcal{L} & & \end{array} \tag{1}$$

where $\mathcal{L} \subset \mathbb{R}^k = \mathbb{R}^n \times \mathbb{R}^{k-n}$ is a lattice and

$$\begin{array}{ll} \pi_1 : \mathbb{R}^k \longrightarrow \mathbb{R}^n & \pi_1(x_1, x_2, \ldots, x_k) = (x_1, x_2, \ldots, x_n) \\ \pi_2 : \mathbb{R}^k \longrightarrow \mathbb{R}^{k-n} & \pi_2(x_1, x_2, \ldots, x_k) = (x_{n+1}, x_{n+2}, \ldots, x_k) \end{array} \tag{2}$$

is called a *cut and project scheme* [1, 2, 7, 10] if the following two conditions are satisfied:

(i) π_1 restricted to $\mathcal{L}$ is injective;
(ii) $\pi_2(\mathcal{L})$ is dense in $\mathbb{R}^{k-n}$.

For any compact set $\Omega \subset \mathbb{R}^{k-n}$ such that

$$\Omega = \overline{\mathrm{int}(\Omega)} \neq \emptyset \tag{3}$$

we define the pattern

$$\Lambda(\Omega) = \{\pi_1(x) \mid x \in \mathcal{L}, \ \ \pi_2(x) \in \Omega\} \tag{4}$$

called a *model set* [1, 2, 7, 9, 10]. Using the mapping

$$L \longrightarrow \mathbb{R}^{k-n} : x \mapsto x^* = \pi_2((\pi_{1|_\mathcal{L}})^{-1}(x)) \tag{5}$$

where $L = \pi_1(\mathcal{L})$, we get

$$\mathcal{L} = \{ (x, x^*) \mid x \in L \} \tag{6}$$

$$\Lambda(\Omega) = \{ x \in L \mid x^* \in \Omega \}. \tag{7}$$

A *self-similarity* of $\Lambda(\Omega)$ is an affine linear mapping [1]

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n : x \mapsto Ax = \lambda Rx + v \tag{8}$$

where $\lambda \in \mathbb{R} - \{0\}$, $v \in \mathbb{R}^n$, and $\mathbb{R}^n \longrightarrow \mathbb{R}^n : x \mapsto Rx$ is an orthogonal transformation such that

$$x \in \Lambda(\Omega) \quad \Longrightarrow \quad Ax \in \Lambda(\Omega) \tag{9}$$

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that is,

$$\left.\begin{array}{r} x \in L \\ x^* \in \Omega \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} Ax \in L \\ (Ax)^* \in \Omega. \end{array}\right. \tag{10}$$

Some examples of model sets and self-similarities can be found in [1, 3–6, 8, 11]. The purpose of this letter is to present a method which allows us to determine certain self-similarities of model sets. Our method is powerful [4–6], but, generally, it does not allow us to find all the self-similarities of a model set [8].

We shall regard $\mathbb{R}^n$ and $\mathbb{R}^{k-n}$ as subspaces of $\mathbb{R}^k$

$$\begin{array}{l} \mathbb{R}^n \equiv \{(x_1, x_2, \ldots, x_n, 0, \ldots, 0) \mid (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\} \\ \mathbb{R}^{k-n} \equiv \{(0, \ldots, 0, x_1, x_2, \ldots, x_{k-n}) \mid (x_1, x_2, \ldots, x_{k-n}) \in \mathbb{R}^{k-n}\} \end{array} \tag{11}$$

and we shall identify π_1 and π_2 with the mappings

$$\begin{array}{ll} \pi_1 : \mathbb{R}^k \longrightarrow \mathbb{R}^k & \pi_1(x_1, x_2, \ldots, x_k) = (x_1, x_2, \ldots, x_n, 0, \ldots, 0) \\ \pi_2 : \mathbb{R}^k \longrightarrow \mathbb{R}^k & \pi_2(x_1, x_2, \ldots, x_k) = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots, x_k). \end{array} \tag{12}$$

Let $\{e_1, e_2, \ldots, e_k\}$ be a basis of $\mathbb{R}^k$ such that

$$\mathcal{L} = \sum_{j=1}^{k} \mathbb{Z} e_j \tag{13}$$

and let

$$\pi_1 = \begin{pmatrix} p_{11} & p_{12} & \ldots & p_{1k} \\ p_{21} & p_{22} & \ldots & p_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ p_{k1} & p_{k2} & \ldots & p_{kk} \end{pmatrix} \qquad \pi_2 = \begin{pmatrix} q_{11} & q_{12} & \ldots & q_{1k} \\ q_{21} & q_{22} & \ldots & q_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ q_{k1} & q_{k2} & \ldots & q_{kk} \end{pmatrix} \tag{14}$$

be the matrices of π_1 and π_2 expressed in terms of the basis $\{e_1, e_2, \ldots, e_k\}$, that is,

$$\pi_1 e_j = \sum_{i=1}^{k} p_{ij} e_i \qquad \pi_2 e_j = \sum_{i=1}^{k} q_{ij} e_i \tag{15}$$

for any $j \in \{1, 2, \ldots, k\}$.

Theorem. *If $\lambda \in \mathbb{R} - \{0\}$, $\lambda' \in [-1, 1]$, and $v \in L$ are such that the entries of the matrix*

$$M = \lambda \pi_1 + \lambda' \pi_2 \tag{16}$$

are integers and

$$\lambda' \Omega + v^* \subset \Omega \tag{17}$$

then

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n : \; x \mapsto Ax = \lambda x + v$$

is a self-similarity of the model set $\Lambda(\Omega)$.

Proof. If the entries $\lambda p_{ij} + \lambda' q_{ij}$ of M are integers then

$$\sum_{j=1}^{k} \alpha_j e_j \in \mathcal{L} \quad \Longrightarrow \quad M\left(\sum_{j=1}^{k} \alpha_j e_j\right) = \sum_{i=1}^{k} \left(\sum_{j=1}^{k} \alpha_j (\lambda p_{ij} + \lambda' q_{ij})\right) e_i \in \mathcal{L}$$

whence

$$(x, x^*) \in \mathcal{L} \quad \Longrightarrow \quad (\lambda x + v, \lambda' x^* + v^*) = M(x, x^*) + (v, v^*) \in \mathcal{L}.$$

Since $\lambda'\Omega + v^* \subset \Omega$ we get

$$\left.\begin{array}{r} x \in L \\ x^* \in \Omega \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} \lambda x + v \in L \\ (\lambda x + v)^* = \lambda' x^* + v^* \in \Omega . \end{array}\right.$$

that is,

$$x \in \Lambda(\Omega) \quad \Longrightarrow \quad \lambda x + v \in \Lambda(\Omega).$$

□

In order to illustrate our method we present an example. Let $\tau = (1+\sqrt{5})/2$ and let $\sigma = (1-\sqrt{5})/2$. We consider the cut and project scheme [3]

$$\begin{array}{ccccc} \mathbb{R}^2 & \xleftarrow{\pi_1} & \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\pi_2} & \mathbb{R}^2 \\ & & \cup & & \\ & & \mathcal{L} & & \end{array} \tag{18}$$

where the lattice

$$\mathcal{L} = \{(\alpha_1 + \alpha_2\tau + (\alpha_3 + \alpha_4\tau)\mathrm{e}^{\pi\mathrm{i}/5}, \alpha_1 + \alpha_2\sigma + (\alpha_3 + \alpha_4\sigma)\mathrm{e}^{3\pi\mathrm{i}/5}) \mid \alpha_j \in \mathbb{Z}\} \tag{19}$$

is defined by using the usual identification $\mathbb{R}^2 \longrightarrow \mathbb{C} : (a, b) \mapsto a + b\mathrm{i}$. In this case

$$L = \{\, \alpha_1 + \alpha_2\tau + (\alpha_3 + \alpha_4\tau)\mathrm{e}^{\pi\mathrm{i}/5} \mid \ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^4 \,\} \tag{20}$$

and

$$(\alpha_1 + \alpha_2\tau + (\alpha_3 + \alpha_4\tau)\mathrm{e}^{\pi\mathrm{i}/5})^* = \alpha_1 + \alpha_2\sigma + (\alpha_3 + \alpha_4\sigma)\mathrm{e}^{3\pi\mathrm{i}/5}. \tag{21}$$

Let $r \in (0, \infty)$ and let

$$\Lambda_r = \left\{ \alpha_1 + \alpha_2\tau + (\alpha_3 + \alpha_4\tau)\mathrm{e}^{\pi\mathrm{i}/5} \ \left| \ \begin{array}{l} (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^4 \\ |\alpha_1 + \alpha_2\sigma + (\alpha_3 + \alpha_4\sigma)\mathrm{e}^{3\pi\mathrm{i}/5}| \leqslant r \end{array} \right. \right\} \tag{22}$$

be the model set defined by using the window

$$\Omega = \{\, y \in \mathbb{R}^2 \mid \ |y| \leqslant r \,\}. \tag{23}$$

One can see that

$$\mathcal{L} = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4 \tag{24}$$

where

$$\begin{aligned} e_1 &= (1, 0, 1, 0) \\ e_2 &= (\tau, 0, \sigma, 0) \\ e_3 &= (\cos(\pi/5), \sin(\pi/5), \cos(3\pi/5), \sin(3\pi/5)) \\ e_4 &= (\tau\cos(\pi/5), \tau\sin(\pi/5), \sigma\cos(3\pi/5), \sigma\sin(3\pi/5)). \end{aligned} \tag{25}$$

and

$$\pi_1 = \mathcal{M}\left(\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \qquad \pi_2 = \mathcal{M}\left(\frac{\sqrt{5}+1}{2\sqrt{5}}, \frac{\sqrt{5}-1}{2\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) \tag{26}$$

where

$$\mathcal{M}(a, b, c) = \begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c & b \end{pmatrix}. \tag{27}$$

The matrix

$$M = \lambda\pi_1 + \lambda'\pi_2 = \mathcal{M}(\alpha, \beta, \gamma)$$

has integer entries if and only if

$$\begin{aligned} \alpha &= \frac{\sqrt{5}-1}{2\sqrt{5}}\lambda + \frac{\sqrt{5}+1}{2\sqrt{5}}\lambda' \in \mathbb{Z} \\ \beta &= \frac{\sqrt{5}+1}{2\sqrt{5}}\lambda + \frac{\sqrt{5}-1}{2\sqrt{5}}\lambda' \in \mathbb{Z} \\ \gamma &= \frac{1}{\sqrt{5}}\lambda - \frac{1}{\sqrt{5}}\lambda' \in \mathbb{Z} \end{aligned} \tag{28}$$

whence

$$\begin{aligned} \lambda &= \frac{\beta+\alpha}{2} + \frac{\beta-\alpha}{2}\sqrt{5} \\ \lambda' &= \frac{\beta+\alpha}{2} - \frac{\beta-\alpha}{2}\sqrt{5} \\ \gamma &= \beta - \alpha. \end{aligned} \tag{29}$$

In view of the theorem, for any $\lambda = p + q\tau \neq 0$ belonging to the set

$$\left\{ \frac{\beta+\alpha}{2} + \frac{\beta-\alpha}{2}\sqrt{5} \;\middle|\; \alpha, \beta \in \mathbb{Z},\ \left| \frac{\beta+\alpha}{2} - \frac{\beta-\alpha}{2}\sqrt{5} \right| \leqslant 1 \right\} = \{\, p + q\tau \mid p, q \in \mathbb{Z},\ |p + q\sigma| \leqslant 1 \,\} \tag{30}$$

and for any $v \in L$ such that $|v^*| \leqslant r - |p + q\sigma| r$, the transformation

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \ x \mapsto \lambda x + v$$

is a self-similarity of the model set Λ_r. More than that, since [3]

$$e^{2\pi i/5} = -1 + \tau e^{\pi i/5} \qquad |(e^{2\pi i/5} x)^*| = |e^{6\pi i/5} x^*| = |x^*| \tag{31}$$

for any $x \in L$, it follows that Λ_r admits the self-similarity

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \ x \mapsto Ax = \lambda \begin{pmatrix} \cos(2\pi/5) & -\sin(2\pi/5) \\ \sin(2\pi/5) & \cos(2\pi/5) \end{pmatrix} x + v. \tag{32}$$

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